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GENERIC DIFFEOMORPHISM WITH SHADOWING PROPERTY ON TRANSITIVE SETS

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Abstract. Let $f : M \to M$ be a diffeomorphism on a closed C^∞ manifold. Let Λ be a transitive set. In this paper, we show that (i) C^1-generically, f has the shadowing property on a locally maximal Λ if and only if Λ is hyperbolic, (ii) f has the C^1-stably shadowing property on Λ if and only if Λ is hyperbolic.

1. Introduction

Let M be a closed C^∞ manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in$ Diff(M). For δ >0, a sequence of points $\{x_i\}_{i=a}^b \subset M$ $(-\infty \le a < b \le \infty)$ is called a *δ-pseudo-orbit* of $f \in$ Diff(M) if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. For a closed f-invariant set $\Lambda \subset M$, we say that f has the *shadowing property* (or Λ is *shadowable* for f) if for every $\varepsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is a $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. In this case, $\{x_i\}_{i=a}^b$ is said to be *ϵ-shadowed* by the point y. Note that only δ-pseudo orbits of f contained in Λ are allowed to be ϵ-shadowed, but the shadowing point $y \in M$ is not necessarily contained in Λ. The notion of the pseudo-orbits very often appears in several branches of the modern theory of dynamical system. For instance, the pseudo-orbit tracing property (shadowing property) usually plays an important role in stability theory([6]).

Given $f \in$ Diff(M), a closed f-invariant set $\Lambda \subset M$ is said to be *chain transitive* if for any points $x, y \in \Lambda$ and $\delta > 0$, there exists a δ-pseudo

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orbit $\{x_i\}_{i=0}^{n} \subset \Lambda (n > 1)$ of f such that $x_0 = x$ and $x_n = y$. A a closed f-invariant set $\Lambda \subset M$ is said to be *transitive* if there is a point $x \in \Lambda$ such that the ω-limit set $\omega(x) = \Lambda$. Note that by definition, transitive sets are chain transitive sets, but the converse is not true([2]). We say that Λ is *isolated (or locally maximal)* if there is an open neighborhood V of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$$

A closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \le C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. Moreover, we say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

We say that a subset $\mathcal{R} \subset \text{Diff}(M)$ is *residual* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case, $\mathcal{R}$ is dense in Diff(M). A property (P) is said to be (C^1)-*generic* if (P) holds for all diffeomorphisms which belong to some residual subset $\mathcal{R}$ of Diff(M).

Study of dynamical systems under C^1-generic condition is very useful. Pugh's closing lemma implies that any transitive set Λ of a C^1-generic diffeomorphism f is the Hausdorff limit of a sequence of periodic orbits P_n of f: i.e., $\lim_{n\to\infty} P_n = \Lambda$. Furthermore, [2] showed that C^1-generically, nontrivial chain transitive sets are approximated in the Hausdorff topology by periodic orbits. In [1], Abdenur and Díaz obtained a necessary and sufficient condition for an isolated transitive set Λ of a C^1-generic diffeomorphism f to be hyperbolic. They have shown that either Λ is hyperbolic, or there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood V of Λ respectively such that every $g \in \mathcal{U}(f)$ does not have the shadowing property on the neighborhood V. Very recently, Lee and Wen ([4]) proved that C^1-generically, isolated chain transitive set is shadowable if and only if it is hyperbolic. In this paper, we study the hyperbolicity of shadowable transitive sets of C^1-generic diffeomorphisms and prove the following.

Theorem 1.1. *An isolated transitive set of a C^1-generic diffeomorphism is hyperbolic if and only if it is shadowable.*

The notion of the C^1-stably shadowing property was introduced in [3]. Let Λ be a closed f-invariant set. We say that f has the C^1-*stably shadowing property* on Λ, if there are C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$(locally maximal), and for any $g \in \mathcal{U}(f)$, g has the shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. In [3, 8], the authors proved that if a diffeomorphism f has the C^1-stably shadowing property on a closed invariant set(chain components, chain transitive sets) then it is hyperbolic.

In this paper we study hyperbolicity of transitive sets under C^1-stably shadowing property and prove the following.

THEOREM 1.2. *Let Λ be a transitive set of f. Then f has the C^1-stably shadowing property on Λ if and only if Λ is hyperbolic.*

2. Proof of Theorem1.1

First, we state some results which will be used in the proof of Theorem 1.1. The following proposition is very useful to prove Theorem 1.1. Let Λ be a transitive set.

PROPOSITION 2.1. *There is a residual set $\mathcal{R}' \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{R}'$, if f has the shadowing property, then for any hyperbolic periodic points $p, q \in \Lambda$*

$$\mathrm{index}(p) = \mathrm{index}(q),$$

where $\mathrm{index}(p) = \dim W^s(p)$.

To prove Proposition 2.1, we need the following lemmas. The following is due to Kupka-Smale Theorem.

LEMMA 2.2. *There is a residual set $\mathcal{R}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{R}_1$, any $p \in P(f)$ is hyperbolic. Further, for any $p, q \in P(f)$, we have $W^s(p) \pitchfork W^u(q) \neq \phi$, and $W^u(p) \pitchfork W^s(q) \neq \phi$, where $P(f)$ is the set of the periodic points of f.*

LEMMA 2.3. *Let $f \in \mathcal{R}_1$. Then for any $p, q \in P_h(f)$,*

$$W^s(p) \pitchfork W^u(q) \neq \phi \quad \text{and} \quad W^u(p) \pitchfork W^s(q) \neq \phi,$$

where $P_h(f)$ is the set of hyperbolic periodic points of f.

Proof. Let $p, q \in \Lambda \cap P_h(f)$ and let $\epsilon(p) > 0$, and $\epsilon(q) > 0$ be small constants such that the local stable manifolds $W^s_{\epsilon(p)}(p)$ and $W^s_{\epsilon(q)}(q)$ and the local unstable manifolds $W^u_{\epsilon(p)}(p)$ and $W^u_{\epsilon(q)}(q)$ respectively are well defined. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. Let $\delta > 0$ be such that every δ-pseudo orbit of f is ϵ-shadowed in Λ. To simplify, $f(p) = p$ and $f(q) = q$. Then there exists $x \in \Lambda$ such that $\omega(x) = \Lambda$ and there exist $n_1 > 0$ and $n_2 > 0$ such that $f^{n_1}(x) \in B_\delta(p)$ and $f^{n_2}(x) \in B_\delta(q)$ with $n_2 > n_1$, where $B_\delta(x)$ is a δ-neighborhood of x. Let $n_1 + l = n_2$ for some $l > 0$. Then we can get a δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ as follows: Put, $x_i = p$, for all $i \le 0$, $f^{n_1+i}(x) = x_i$ for $1 \le i \le l-1$, and $x_i = q$ for $i \ge l$. Thus

$$\begin{aligned}\xi =&\{\ldots,\ p,\ p,\ f^{n_1}(x), \ldots,\ f^{n_2-1}(x),\ q,\ q, \ldots\}\\ =&\{\ldots, x_{-1}, x_0(=p), x_1, \ldots, x_{l-1}, x_l(=q), \ldots\}\end{aligned}$$

is a δ-pseudo orbit of f. By the shadowing property, there exists $y \in B_\varepsilon(p)$ such that $d(f^i y, p) < \varepsilon$, for all $i \le 0$ and $d(f^i y, q) < \varepsilon$ for all $i \ge l$. Thus $f^i(y) \in W^u_\varepsilon(p)$ for all $i \le 0$ and $f^i(y) \in W^s_\varepsilon(p)$ for all $i \ge l$. This implies $y \in W^u(p) \cap W^s(q)$ proving $W^s(p) \cap W^u(q) \ne \emptyset$. This completes the proof of Lemma 2.3. □

Let p and q be a hyperbolic periodic points of f. We say that p and q are *homoclinically related* and write $p \sim q$ if $W^s(p)$ (respectively, $W^u(p)$) and $W^u(q)$ (respectively, $W^u(q)$) have nonempty transverse intersections. It is clear that if $p \sim q$, then $\text{index}(p) = \text{index}(q)$, where $\text{index}(p)$ is the index of p, namely, the dimension of the stable eigenspace E^s_p of p.

Proof of Proposition 2.1. Form the above, we show that $p \sim q$. Let $\mathcal{R}' = \mathcal{R}_1$, and let p and q be a hyperbolic periodic points in Λ. For $f \in \mathcal{R}'$, assume that f has the shadowing property on Λ. Then by Lemma 2.3, $W^s(p) \cap W^u(q) \ne \emptyset$. Since $f \in \mathcal{R}_1$, $W^s(p) \pitchfork W^u(q) \ne \emptyset$ and $W^u(p) \pitchfork W^s(q) \ne \emptyset$. Thus $\text{index}(p) = \text{index}(q)$. □

The following lemma can be obtained by using Pugh's closing lemma and the notion of transitivity.

LEMMA 2.4. *Let Λ be a nontrivial transitive set. Then there are a sequence $\{g_n\}_{n\in\digamma^+}$ of diffeomorphisms and a sequence $\{P_n\}$ of periodic orbits of g_n with period $\pi(P_n) \to \infty$ such that $g_n \to f$ in the C^1-topology and $P_n \to_H \Lambda$ as $n \to \infty$, where $\to_H$ is the Hausdorff limit, and $\pi(P_n)$ is the period of P_n.*

Proof. See [10, Corollary 2.7.1]. □

LEMMA 2.5. *There is a residual set $\mathcal{R}_2 \subset$ Diff(M) such that every $f \in \mathcal{R}_2$ satisfies the following property : For any closed f-invariant set*

$\Lambda \subset M$, *if there are a sequence of diffeomorphisms* f_n *conversing to* f *and a sequence of hyperbolic periodic orbits* P_n *of* f_n *with index* k *verifying* $lim_{n\to\infty}P_n = \Lambda$, *then there is a sequence of hyperbolic periodic orbits* Q_n *of* f *with index* k *such that* Λ *is the Hausdorff limit of* Q_n.

Proof. See [4, Lemma 2.2] □

We say that a point x in M is *well closable* for $f \in \mathrm{Diff}(M)$ if for any $\varepsilon > 0$, there are $g \in \mathrm{Diff}(M)$ with $d_{C^1}(g, f) < \varepsilon$ and a periodic point p of g such that $d(f^n(x), g^n(p)) < \varepsilon$ for all $0 \leq n \leq \pi(p)$, where $\pi(p)$ is the period of p, and d_{C^1} is the usual C^1-metric. Let $\mu(\Sigma_f) = 1$ denote the set of well closable points of f. Mane's ergodic closing lemma [4] says that $\mu(\Sigma_f) = 1$ for any f-invariant Borel probability measure μ on M.

Let $\mathcal{M}$ be the space of all Borel measures μ on M with the weak* topology. It is easy to check that, for any ergodic measure $\mu \in \mathcal{M}$ of f, μ is supported on a periodic orbit $P = \{p, f(p), \cdots, f^{\pi(p)-1}(p)\}$ of f if and only if $\mu = \frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \delta_{f^i(p)}$, where δ_x is the atomic measure respecting x. The following lemma comes from the Mane's ergodic closing lemma which gives the measure theoretical viewpoint on the approximation by periodic orbits.

Lemma 2.6. *There is a residual set* $\mathcal{R}_3 \subset$ *Diff*(M) *such that every* $f \in \mathcal{R}_3$ *satisfies the following property : Any ergodic invariant measures supported by periodic orbits* P_n *of* f *in the weak** *topology. Moreover, the orbits* P_n *converges to the support of* μ *in the Hausdorff topology.*

Proof. See [4, Lemma 2.3] □

Now we define the residual subset $\mathcal{R}$ of $\mathrm{Diff}(M)$ required in the statement of Theorem 1.1 as follows: $\mathcal{R} = \mathcal{R}_1 (= \mathcal{R}') \cap \mathcal{R}_2 \cap \mathcal{R}_3$. Then we have the following proposition which is crucial to prove Theorem 1.1.

Proposition 2.7. *Let* $f \in \mathcal{R}$, *and let* Λ *be a shadowable transitive set of* f *which is isolated. Then there exist constants* $m > 0$ *and* $\lambda \in (0, 1)$ *such that for any periodic point* $p \in \Lambda$,

$$\prod_{i=0}^{\pi(p)-1} \|Df^m|_{E^s(f^{im}(p))}| \leq \lambda^{\pi}(p),$$

$$\prod_{i=0}^{\pi(p)-1} \|Df^{-m}|_{E^u(f^{-im}(p))}\| \leq \lambda^{\pi}(p), \text{ and}$$

$$\|Df^m|_{E^s(p)}\| \cdot \|Df^{-m}|_{E^u(f^m(p))}\| < \lambda^2.$$

where $\pi(p)$ denote the period of p.

Proof. See [4, Propositon 2.1] □

End of the proof of Theorem 1.1. By Lemma 2.1 and the third property of Proposition 2.7, we can see that Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ which satisfies $E(p) = E^s(p)$ and $F(p) = E^u(p)$ for every periodic point $p \in \Lambda$. To complete the proof of Theorem 1.1., it is enough to show that Df^m is contracting on E and Df^m is expanding on F if Λ is shadowable for f. Suppose Df^m is not contracting on E. Then, we can find a point $b \in \Lambda$ such that

$$\prod_{i=0}^{k-1} \|Df^m|_{E(f^{im}(b))}\| \geq 1$$

for any $k > 0$. Denote by δ_x the atomic measure respecting x. Let us consider a sequence $\{1/n \sum_{i=0}^{n-1} \delta_{f^{im}(b)} : n \in \mathbb{Z}^+\}$ in $\mathcal{M}$, and take an accumulation point $\mu \in \mathcal{M}$ of the sequence. Then we can see that μ is a f^m-invariant probability measure on M supported on Λ which satisfies

$$\int \log(\|Df^m|_{E(x)}\|)d\mu \geq 0.$$

Note here that we can extend E continuously to the whole manifold M. By the ergodic decomposition theorem, there is an ergodic measure μ_0 supported on Λ such that

$$\int \log(\|Df^m|_{E(x)}\|)d\mu_0 \geq 0.$$

Then, by Lemma 2.3, we can take a sequence of ergodic f^m-invariant measures μ_n such that the support of each μ_n is a periodic orbit P_n of f, $\{\mu_n\}$ converges to μ_0 and $\{P_n\}$ converges to Λ. Since Λ is isolated, we may assume that every P_n is contained in Λ for sufficiently large n. If we apply Proposition 2.6, then we have

$$\int \log(\|Df^m|_{E(x)}\|)d\mu_n < \log \lambda$$

for sufficiently large n. Since μ_n converges to μ_0 in the weak* topology, we have

$$\int \log(\|Df^m|_{E(x)}\|)d\mu_n \to \int \log(\|Df^m|_{E(x)}\|)d\mu_0$$

as $n \to \infty$. Hence we get $\int \log(\|Df^m|_{E(x)}\|)d\mu_0 < 0$. This is a contradiction. Thus Df^m is contracting on E. Similarly we can show that

Df^m is expanding on F. □

3. Proof of Theorem 1.2

Let M be as before, and let $f \in \mathrm{Diff}(M)$. The following lemma was proved by [3].

LEMMA 3.1. *Suppose that f has the C^1-stably shadowing property on Λ. Then there exists a C^1-neighborhood $\mathcal{V}(f) \subset \mathcal{U}(f)$ of f such that for each $g \in \mathcal{V}(f)$, every $q \in \Lambda_g \cap P(g)$ is hyperbolic.*

Let Λ be a transitive set of f and suppose f has the C^1-stably shadowing property on Λ. Then by Lemma 3.1, we can see the periodic points are dense in Λ as follows.

LEMMA 3.2. $\Lambda = \overline{\Lambda \cap P(f)}$.

Proof. See [8, Lemma 4] □

LEMMA 3.3. *Let $\mathcal{V}(f)$ be given by Lemma 3.1. Then there are constants $K > 0$, $m > 0$ and $0 < \lambda < 1$ such that :*

(a) for any $g \in \mathcal{V}(f)$, if $q \in \Lambda_g \cap P(g)$ has minimum period $\pi(q) \geq m$, then

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)} g^m_{|E^s_{g^{im}(q)}}\| < K\lambda^k$$

and

$$\prod_{i=0}^{k-1} \|D_{g^{-im}(q)} g^{-m}_{|E^u_{g^{-im}(q)}}\| < K\lambda^k$$

where $k = [\pi(q)/m]$.
(b) Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ with $dimE = index(p)$.

However, if Λ be a transitive set of f and f has the C^1-stably shadowing property on Λ, then such case cannot happen. Hence we are needed the following.

Let $H_f(p)$ be the homoclinic class of p ; that is, the closure of the set of all $q \in P(f)$(denote, the set of periodic points of f by $P(f)$) such that $p \sim q$. Note that, by the well-known Smale's transeverse homoclinic point theorem, $H_f(p)$ coincides with the closure of the set of all "transeverse" homoclinic points $x \in W^s(p) \cap W^u(p)$ of p. Denote by $\mathcal{O}_f(p)$ the periodic f-orbit of p, and set

$$H_f(\mathcal{O}_f(p)) = H_f(p) \cup \cdots \cup H_f(f^{\pi(p)-1}(p)).$$

LEMMA 3.4. *$\Lambda \subset H_f(\mathcal{O}_f(p)$ for some $p \in P(f)$ and $\Lambda = \overline{P_j(f|_\Lambda)}$.*

Proof. Let U be a compact neighborhood of Λ where Λ is isolated. Fix any $0 < \varepsilon < \varepsilon(p)$, and let $\delta = \delta(\varepsilon) > 0$ be the number of the shadowing property of $f|_\Lambda$ for ε. Since $f|_\Lambda$ is transitive, for any $x \in \Lambda$, there are $y \in B_\delta(p)$ and $0 < l_1 < l_2$ such that $f^{l_1}(y) \in B_\varepsilon(y)$ and $f^{l_2}(y) \in B_\delta(p)$. Put $y_{-i} = f^{-i}(p)$ for all $i \geq 0$, $y_i = f^i(y)$ for all $0 \leq i \leq l_2$ and $y_i = f^{i-l_2}(p)$ for all $i \geq i_2$. Then $\{y_i\}_{i \in Z} \subset \Lambda$ is a δ-pseudo orbit of f. There exists $z \in \Lambda$, ε-nearby y, ε-shadowing the pseudo-orbit, and thus

$$z \in (W^s_{\varepsilon(p)}(\mathcal{O}_f(p)) \cap W^u_{\varepsilon(p)}(\mathcal{O}_f(p))) \cap B_\varepsilon(y) \neq \emptyset$$

Moreover, since $f|_\Lambda$ is C^1-stably shadowing, z is a transverse intersection point of $W^s_{\varepsilon(p)}(\mathcal{O}_f(p))$ and $W^u_{\varepsilon(p)}(\mathcal{O}_f(p))$. Indeed, if z is a non-transverse intersection point, with a C^1-small perturbation of the map f in U, we can construct g, C^1-nearby f, possessing a small affine piece containing z whose whole g-orbit is also contained in U. From this, it is possible to construct a non-shadowable pseudo-orbit of g in $\Lambda_g(U)$. This is a contradiction because g is C^1-nearby f so that $g|_{\Lambda_g(U)}$ has the shadowing property. Since ε is arbitral, $y \in H_f(\mathcal{O}_f(p))$ is concluded, and thus $\Lambda \subset H_f(\mathcal{O}_f(p))$.
Now we show that, $\Lambda = \overline{P_j(f|_\Lambda)}$. By the well-known Smale's transverse homoclinic point theorem, there is a saddle $q \in P(f)$ nearby z such that $q \sim p$. Observe that by construction of the saddle q, we see that $\mathcal{O}_f(q)$ is contained in a small neighborhood of the homoclinic orbit $\mathcal{O}_f(z)$ of z so that $q \in \Lambda$ since Λ is locally maximal in U. Therefore, the set of hyperbolic periodic points $q \in P(f)$ with index(q) =index(p) is dense in Λ. Then the proof is complete. □

It is known that a non-hyperbolic homoclinic class $H_f(\mathcal{O}_f(p))$ contains saddle periodic points with different indices in general. Thus the chain transitive set Λ may contain saddle periodic points with different indices in general. However, such case cannot happen in our setting.

LEMMA 3.5. *Under the same notation and assumption of Lemma 3.4, we have $q \sim p$ for any $q \in \Lambda \cap P(f)$.*

Proof. If Lemma 3.5 is false, then there exists $q \in \Lambda \cap P(f)$ such that index$(q) \neq$index(p). Since $f|_\Lambda$ is transitive and both p , q are hyperbolic, there is the so-called heterodimensional cycle between p and q in Λ. More precisely, there are $x \in W^u(q) \cap W^s(p)$ and $y \in W^s(q) \cap W^u(p)$ such that $\dim W^u(q) + \dim W^s(p) < dim M$ or $\dim W^s(q) + \dim W^u(p) < \dim M$. Observe that $x, y \in \Lambda$, with a C^1-small perturbation of the map f in U, we can construct $g \in \mathcal{U}(f)$ possessing a small affine piece containing y whose whole g-orbit is also contained in U. From this fact, it is possible to construct a non-shadowable pseudo-orbit of g in $\Lambda_g(U)$ since Λ has a dominated splitting. This is a contradiction, since $g|_{\Lambda_{g(U)}}$ has the shadowable property. □

Let us recall Mañé's Ergodic Closing Lemma, for any f-invariant measure μ.Then we have $\mu(\Sigma_f) = 1$.

End of the proof of Theorem 1.2. Suppose that f as before, and let U be a compact neighborhood of Λ as in the isolated of it. We show "only if " part (see the paragraph following the statement of Theorem 1.2). By Lemma 3.4, $\Lambda = \Lambda_j(f)$, where $0 < j = \text{index}(p) < \dim M$. Let $\mathcal{V}(f)$ be the C^1-neighborhood of f given by $\Lambda_i(f) = \overline{P_i(f|_\Lambda)} = \emptyset$ if $i \neq j$. Let $\widetilde{U}(f) \subset \mathcal{V}(f)$ be a small connected C^1-neighborhood of f. If $g \in \widetilde{U}(f)$ satisfy $g = f$ on $M \backslash U_j$, then index(q) = index(p) for any $q \in \Lambda_g \cap P(g)$. Now, Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that $\dim E$ = index(p). Then we can show that

$$\liminf_{n\to\infty} \|D_x f^n_{|E_x}\| = 0$$

and

$$\liminf_{n\to\infty} \|D_x f^{-n}_{|F_x}\| = 0$$

for all $x \in \Lambda$, and thus, the splitting is hyperbolic.
Let us display a brief outline of the proof of the first assertion (the second is similar). Indeed, let $\varphi(x) = \log \|D_x f^m_{|E_x}\|$ for $x \in \Lambda$. If $\liminf_{n\to\infty} \|D_x f^n_{|E_x}\| = 0$ does not hold for all $x \in \Lambda$, then there are subsequence $\{j_n\}_{n\in N}$ and an f^m-invariant probability measure μ on Λ such that

$$\int_\Lambda \varphi d\mu = \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log \|D_{f^{mi}(x)} f^m_{|E_{f^{mi}(x)}}\| \geq 0$$

By Birkhoff's theorem, together with Mañé's Ergodic Closing Lemma, we can find $z \in \Lambda \cap \Sigma_f$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{mi}(z)} f^m_{|E_{f^{mi}(z)}}\| \geq 0$$

Here Σ_f is the set of Mañé's Ergodic Closing Lemma. By Lemma 3.3 (a), we see $z \notin P(f)$. Let $K > 0,\ m > 0$ and $0 < \lambda < 1$ given by Lemma 3.3, and take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{mi}(z)} f^m_{|E_{f^{mi}(z)}}\| \geq \log \lambda_0$$

when $n > n_0$. Then, by Mañé's Ergodic Closing Lemma we can find $\overline{g} \in \widetilde{\mathcal{U}}_0(f)(\overline{g} = f$ on $M \setminus U_j)$ and $\overline{z} \in \Lambda_{\overline{g}} \cap P(\overline{g})$ nearby z. Moreover, $\text{index}(\overline{z}) = \text{index}(p)$, by applying the so-called Franks' lemma we can construct $\hat{g} \in \widetilde{\mathcal{U}} \subset \mathcal{V}(f)$ C^1-nearby $\overline{g}$ such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} \|D_{\widetilde{g}^{im}(\overline{z})} \hat{g}^m_{|E^s_{g^{im}(\overline{z})}}\|$$

On the other hand, we see that

$$\prod_{i=0}^{k-1} \|D_{\widetilde{g}^{im}(\overline{z})} \hat{g}^m_{|E^s_{g^{im}(\overline{z})}}\| < K\lambda^k.$$

We can choose the period $\pi(\bar{z})$ $(> n_0)$ of $\bar{z}$ as large as $\lambda_0^k \geq K\lambda^k$. Here $k = [\pi(\bar{z})/m]$. This is a contradiction. Thus, $\liminf_{n\to\infty} \|D_x f^n_{|E_x}\| = 0$ for all $x \in \Lambda$.
We can show the assertion $\liminf_{n\to\infty} \|D_x f^{-n}_{|E_x}\| = 0$ in a similar way for all $x \in \Lambda$, and so Λ is hyperbolic. The theorem is proved. □

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